THE GAUSS MAP OF PSEUDO-ALGEBRAIC MINIMAL SURFACES

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Abstract. We refine Osserman's argument on the exceptional values of the Gauss map of algebraic minimal surfaces. This gives an effective estimate for the number of exceptional values and the totally ramified value number for a wider class of complete minimal surfaces that includes algebraic minimal surfaces. It also provides a new proof of Fujimoto's theorem for this class, which not only simplifies the proof but also reveals the geometric meaning behind it.

1. Introduction

The problem of finding the maximal number D_g of the exceptional values of the Gauss map g of a complete non-flat minimal surface M in $\mathbb{R}^3$ was settled by Fujimoto [F1], [F2] with the best possible upper bound being 4. Indeed, for any number r, $0 \leq r \leq 4$, we can construct complete minimal surfaces in $\mathbb{R}^3$ whose Gauss map omits exactly r values. Moreover, Fujimoto proved that the totally ramified value number ν_g, which gives more detailed information than D_g does, satisfies $\nu_g \leq 4$, and this inequality is the best possible. Here, $b \in \mathbb{P}^1 = \mathbb{CP}^1$ is called a totally ramified value of $g\colon M \to \mathbb{P}^1$ if at all the inverse image points of b, g branches. The exceptional values are regarded as totally ramified values, since it is natural to consider the multiplicity of an exceptional value to be infinite in the context of the Nevanlinna theory [Ko]. The totally ramified value number ν_g is a weighted sum of the number of totally ramified values (see §3 for a precise definition). In particular, $D_g \leq \nu_g$ holds.

On the other hand, Osserman [O1] proved that the Gauss map of a non-flat algebraic minimal surface omits at most 3 values. By an algebraic minimal surface, we mean a complete minimal surface with finite total curvature. There are no

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known examples, however, of algebraic minimal surfaces whose Gauss map omits 3 values, while there are many examples, of almost all topological types with the Gauss maps omitting 2 values [MS]. Thus an established conjecture is that the sharp upper bound of D_g is 2. Moreover, as in the case of Fujimoto's theorem, we have an implicit conjecture that the same is true for the totally ramified value number ν_g.

Surprisingly, the first author found algebraic minimal surfaces with totally ramified value number $\nu_g = 2.5$, i.e., strictly larger than 2 [Ka]. This overthrew the above implied conjecture, and a qualified problem to consider is then:

Does there exist some κ, $2.5 \leq \kappa < 3$ which is an upper bound $\nu_g \leq \kappa$ for ν_g?

The totally ramified value number as well as the defect is well investigated in the Nevanlinna theory of transcendental meromorphic functions on $\mathbb{C}$. But for the Gauss map of *algebraic* minimal surfaces, this number has not been considered, at the best of the authors' knowledge. Since this is an indispensable number when we discuss the problem of exceptional values, we study it here by refining Osserman's algebraic argument, and the results turned out to be much more effective than we had expected (Theorem 3.3). They are effective in the sense that the upper bound we obtain is described in terms of the degree of the Gauss map and the topological data of M. Moreover, in some sense, this is the best possible result, and gives a proper extension of Osserman's results. In particular, we prove that the totally ramified value number of an algebraic minimal surface is strictly less than 4.

Another advantage of the approach here is that we can develop our arguments on a wider class of complete minimal surfaces, i.e., on all complete minimal surfaces whose Weierstrass data descends to meromorphic data on a compact Riemann surface. We refer to such minimal surfaces as *pseudo-algebraic minimal surfaces* (§3). As we do not assume the period condition, the surfaces may have infinite total curvature. For this class of minimal surfaces, we obtain a new proof of Fujimoto's theorem, which reveals the reason why $\nu_g \leq 4$ holds. We also give a kind of unicity theorem (Theorem 5.1), which asks the least number of values at which if two Gauss maps g_1 and g_2 have the same inverse image then $g_1 = g_2$.

Moreover, our argument suggests how to estimate the characteristic function $T_g(r)$ of the Gauss map lifted to the universal covering surface, which plays an essential role in the Nevanlinna theory on the unit disk (§6). We believe that our results give an important link to the Nevanlinna theory for future research.

2. Preliminaries

A minimal surface $x\colon M \to \mathbb{R}^3$ is originally considered as a surface spanning a given frame with least area. The Euler-Lagrange equation turns out to be

$$\triangle x = 0\,, \tag{1}$$

i.e., each coordinate function of a minimal surface is harmonic. Thus there exist no compact minimal surfaces without boundary. With respect to a complex parameter $z = u + iv$ of the surface, (1) is given by $\bar{\partial}\partial x = 0$ where $\partial = \frac{\partial}{\partial u} - i\frac{\partial}{\partial v}$ following Osserman [O2]. Thus if we put

$$\partial x = (\phi_1, \phi_2, \phi_3)\,,$$

ϕ_j's are holomorphic differentials on M. These satisfy

(C) $\sum \phi_i^2 = 0$: conformality condition
(R) $\sum |\phi_i|^2 > 0$: regularity condtion
(P) For any cycle $\gamma \in H_1(M, \mathbb{Z})$, $\Re \int_\gamma \phi_i = 0$: period condtion

We recover x by the real Abel-Jacobi map (called the Weierstrass-Enneper representation formula)

$$x(z) = \Re \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3) \tag{2}$$

up to translation. From here on we restrict our attention to non-flat minimal surfaces. If we put

$$\begin{cases} hdz = \phi_1 - i\phi_2 \\ g = \dfrac{\phi_3}{\phi_1 - i\phi_2} \end{cases} \tag{3}$$

then hdz is a holomorphic differential and g is a meromorphic function on M. Geometrically, it is well-known that g is the stereographically projected Gauss map of M. We call (hdz, g) the Weierstrass data. This is related to ϕ_j's in a one to one way by

$$\begin{cases} \phi_1 = \dfrac{h}{2}(1 - g^2)dz \\ \phi_2 = \dfrac{ih}{2}(1 + g^2)dz \\ \phi_3 = hgdz\,. \end{cases} \tag{4}$$

If we are given a holomorphic differential hdz and a meromorphic function g on M, we get ϕ_j's by this formula. They satisfy (C) automatically, and the condition (R) is interpreted as the poles of g of order k coincides exactly with the zeros of hdz of order $2k$, because the induced metric on M is given by

$$ds^2 = \frac{|h|^2(1 + |g|^2)^2}{4}|dz|^2.$$

A minimal surface is complete if all divergent paths have infinite length with respect to this metric. In general, for a given meromorphic function g on M, it is not so hard to find a holomorphic differential hdz satisfying (R). But the period condition (P) always causes trouble. When (P) is not satisfied, we anyway obtain a minimal surface on the universal covering surface of M.

Here we notice that the triple of holomorphic differentials

$$e^{i\theta}(\phi_1, \phi_2, \phi_3), \quad \theta \in \mathbb{R}$$

also satisfies (C) and (R). The corresponding Weierstrass data is given by

$$\begin{cases} g^\theta(z) = g(z) \\ h^\theta dz = e^{i\theta} h dz\,. \end{cases} \tag{5}$$

As (P) is scarcely satisfied by these data, we get an S^1 parameter family of minimal surfaces defined on the universal covering surface by (2), which is called the associated family. Note that all surfaces in this family have the same Gauss map.

EXAMPLE 2.1. Catenoid and Helicoid are well-known as surfaces belonging to the same associated family. The Gauss map of this family omits two values.

Now the Gauss curvature K of M is given by

$$K = -\Big(\frac{4|g'|}{|h|(1+|g|^2)^2}\Big)^2 \tag{6}$$

and the total curvature by

$$\tau(M) = \int_M K dA = -\int_M \Big(\frac{2|g'|}{(1+|g|^2)}\Big)^2 du \wedge dv \tag{7}$$

where dA is the surface element of M. Note that $|\tau(M)|$ is the area of M with respect to the (singular) metric induced from the Fubini-Study metric of $\mathbb{P}^1$ by g. When the total curvature of a complete minimal surface is finite, the surface is called *an algebraic minimal surface.*

THEOREM 2.2 (Huber, Osserman). *An algebraic minimal surface* $x : M \to \mathbb{R}^3$ *satisfies:*

(i) M *is conformally equivalent to* $\overline{M} \setminus \{p_1, \dots, p_k\}$ *where* $\overline{M}$ *is a compact Riemann surface, and* $p_1, \dots, p_k \in \overline{M}$ [H].
(ii) *The Weierstrass data* (hdz, g) *is extended meromorphically to* $\overline{M}$ [O1].

We denote the number of exceptional values of g by D_g. Other than Catenoid, there are many examples of algebraic minimal surfaces with $D_g = 2$, which include those of hyperbolic type.

THEOREM 2.3 (Miyaoka-Sato [MS]). *There exist algebraic minimal surfaces with* $D_g = 2$*, for*

(i) $G = 0, k \geq 2$
(ii) $G = 1, k \geq 3$
(iii) $G \geq 2, k \geq 4$

where G *(resp.* k*) is the genus (resp. the number of punctures) of the Riemann surface on which the surfaces are defined.*

When $G = 0$ and $k = 2$, all such minimal surfaces are classified. Examples for $G = 0$ and $k = 3$ given below [MS, Proposition 3.1] are important for later argument: let $M = \mathbb{P}^1 \setminus \{\pm i, \infty\}$, and define a Weierstrass data by

$$\begin{cases} g(z) = \sigma \dfrac{z^2 + 1 + a(t-1)}{z^2 + t} \\ hdz = \dfrac{(z^2+t)^2}{(z^2+1)^2} dz, \quad (a-1)(t-1) \neq 0 \\ \sigma^2 = \dfrac{t+3}{a\{(t-1)a+4\}}\,. \end{cases} \tag{8}$$

For any a, t satisfying $\sigma^2 < 0$, we obtain an algebraic minimal surface of which Gauss map omits two values $\sigma, \sigma a$.

Applying the covering method to this surface (see Remark 3.6), we obtain examples of (ii) and (iii). But as these examples have all the same image in $\mathbb{R}^3$, we further constructed mutually non-congruent examples for $G = 1$ and $k = 4$, by generalizing Costa's surface [MS,Theorem 3]. For details see Remark 4.2.

3. Pseudo-algebraic minimal surfaces and the main results

Definition. We call a complete minimal surface in $\mathbb{R}^3$ *pseudo-algebraic*, if the following conditions are satisfied:

(i) The Weierstrass data (hdz, g) is defined on a Riemann surface $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$, $p_j \in \overline{M}$, where $\overline{M}$ is a compact Riemann surface.
(ii) (hdz, g) can be extended meromorphically to $\overline{M}$.

We call M *the basic domain* of the pseudo-algebraic minimal surface.

Since we do not assume the period condition on M, a pseudo-algebraic minimal surface is defined on some covering surface of M, in the worst case, on the universal covering. Note that Gackstätter called such surfaces *abelian minimal surfaces* [G].

Algebraic minimal surfaces and their associated surfaces are certainly pseudo-algebraic. Another important example is Voss' surface. The Weierstrass data of this surface is defined on $M = \mathbb{C} \setminus \{a_1, a_2, a_3\}$ for distinct $a_1, a_2, a_3 \in \mathbb{C}$, by

$$\begin{cases} g(z) = z \\ hdz = \dfrac{dz}{\Pi_j (z - a_j)}\,. \end{cases} \tag{9}$$

As this data does not satisfy the period condition, we get a minimal surface $x : \mathbb{D} \to \mathbb{R}^3$ on the universal covering disk of M. In particular, it has infinite total curvature. We can see that the surface is complete and the Gauss map omits four values a_1, a_2, a_3, ∞. Starting from $M = \mathbb{C} \setminus \{a_1, a_2\}$, we get similarly a complete minimal surface $x : \mathbb{D} \to \mathbb{R}^3$, of which Gauss map omits three values a_1, a_2, ∞. The completeness restricts the number of points a_j's to be less than four. Note that in both cases, no elements of the associated family satisfy (P), hence have infinite total curvature.

REMARK 3.1. There exist complete minimal surfaces with $D_g = 4$ which are not pseudo-algebraic (see [L]).

Now we define the totally ramified value number ν_g of g.

DEFINITION. We call $b \in \mathbb{P}^1$ a totally ramified value of g when at any inverse image of b, g branches. We regard exceptional values also as totally ramified values. Let $\{a_1, \dots, a_{r_0}, b_1, \dots, b_{l_0}\} \subset \mathbb{P}^1$ be the set of totally ramified values of g, where a_j's are exceptional values. For each a_j, put $\nu_j = \infty$, and for each b_j, define ν_j to be the minimum of the multiplicity of g at points $g^{-1}(b_j)$. Then we have $\nu_j \geq 2$. We call

$$\nu_g = \sum_{a_j, b_j} \left(1 - \frac{1}{\nu_j}\right) = r_0 + \sum_{j=1}^{l_0} \left(1 - \frac{1}{\nu_j}\right)$$

the totally ramified value number of g.

To explain the natural meaning of this number, we need the second main theorem in the Nevanlinna theory, which we have no space to mention here (See [Ko]). Note that though ν_g is a rational number, the upper bound is given by the integer 4 by Fujimoto's theorem.

THEOREM 3.2 (Kawakami [Ka]). *The Gauss map of the algebraic minimal surfaces given in* (8) *has totally ramified value number 2.5.*

In fact, it has two exceptional values, and another totally ramified value at $z = 0$ where $g'(z) = 0$. This theorem is a breakthrough to propose

CONJECTURE. For *algebraic* minimal surfaces, there exists $2.5 \leq \kappa < 3$ which satisfies $\nu_g \leq \kappa$.

Note that the conjecture is *not* for pseudo-algebraic minimal surfaces, since Voss' surfaces satisfy $\nu_g = 3$ and 4. But we develop an algebraic argument on the pseudo-algebraic minimal surfaces in a unified way, and then specialize the results in the algebraic case. The following is the main result of this paper.

THEOREM 3.3. *Consider a non-flat pseudo-algebraic minimal surface with the basic domain $M = \overline{M} \setminus \{p_1, \dots, p_k\}$. Let G be the genus of $\overline{M}$, and let d be the degree of g considered as a map on $\overline{M}$. Then we have*

$$D_g \leq 2 + \frac{2}{R}, \quad \frac{1}{R} = \frac{G - 1 + k/2}{d} \leq 1\,. \tag{10}$$

More precisely, if the number of (not necessarily totally) ramified values other than the exceptional values of g is l, we have

$$D_g \leq 2 + \frac{2}{R} - \frac{l}{d}\,. \tag{11}$$

On the other hand, the totally ramified value number of g satisfies

$$\nu_g \leq 2 + \frac{2}{R}\,. \tag{12}$$

In particular, we have

$$D_g \leq \nu_g \leq 4\,, \tag{13}$$

and for algebraic minimal surfaces, the second inequality is a strict inequality. (11) and (12) are best possible in both algebraic and non-algebraic cases.

The geometric meaning of the ratio R is given in §6. This theorem implies the following known facts:

Corollary 3.4 (Osserman, Fang, Gackstätter). *For algebraic minimal surfaces, we have:*

(i) *When $G = 0$, $D_g \leq 2$ holds.*
(ii) *When $G = 1$ and M has a non-embedded end, $D_g \leq 2$ holds. If $G = 1$ and $D_g = 3$ occur, $d = k$ follows and g does not branch in M, so is a non-branched covering of $\mathbb{P}^1 \setminus \{3$ points$\}$.*

Remark 3.5. Fang [Fa, Theorem 3.1] shows that algebraic minimal surfaces with $d \leq 4$ satisfy $D_g \leq 2$ (see [WX] for $d \leq 3$).

Remark 3.6. There exists a way of construction of algebraic minimal surfaces by a covering method of Klotz-Sario [BC]. Indeed, if $x : M \to \mathbb{R}^3$ is an algebraic minimal surface, and if $\pi : \hat{M} \to M$ is a non-branched covering surface of $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$, then we obtain a new algebraic minimal surface by $\hat{x} = x \circ \pi : \hat{M} \to \mathbb{R}^3$. This surface has the same image as the original one, but the domain $\hat{M}$ has different topological type. Nevertheless, we can see that *the ratio R is invariant* under this construction, via a little algebraic argument. Certainly, D_g and ν_g are also invariant under covering construction.

4. Proof

The proof of Theorem 3.3 is given by a refinement of the proof of Osserman's theorem in [O1]. In order to simplify the argument, we may assume without loss of generality that g is neither zero nor pole at p_j, and moreover, zeros and poles of g are simple. By completeness, hdz has poles of order $\mu_j \geq 1$ at p_j. The period condition implies $\mu_j \geq 2$, but here we do not assume this. Let α_s be (simple) zeros of g, β_t (simple) poles of g. The following table shows the relation between zeros and poles of g, hdz and $ghdz$. The upper index means the order.

z	α_s	β_t	p_j
g	0^1	∞^1	
hdz		0^2	∞^{μ_j}
$ghdz$	0^1	0^1	∞^{μ_j}

Applying the Riemann-Roch formula to the meromorphic differential hdz or $ghdz$ on $\overline{M}$, we obtain

$$2d - \sum_{j=1}^{k} \mu_j = 2G - 2\,.$$

Note that this equality depends on the above setting of zeros and poles of g, though d is an invariant. Thus we get

$$d = G - 1 + \frac{1}{2}\sum_{j=1}^{k} \mu_j \geq G - 1 + \frac{k}{2}\,, \tag{14}$$

and

$$R^{-1} \leq 1\,. \tag{15}$$

When M is an algebraic minimal surface or its associated surface, we have $\mu_j \geq 2$ and so $R^{-1} < 1$.

Now, we prove (11) (and (12)). Assume g omits $r_0 = D_g$ values, and let n_0 be the sum of the branching orders of g at these exceptional values. Moreover, let n_b be the sum of branching orders at the inverse images of non-exceptional (not necessarily totally) ramified values $b_1, \ldots, b_l$ of g. We see

$$k \geq dr_0 - n_0\,, \quad n_b \geq l\,. \tag{16}$$

Let n_g be the total branching order of g. Then applying Riemann-Hurwitz's theorem to the meromorphic function g on $\overline{M}$, we obtain

$$n_g = 2(d + G - 1) = n_0 + n_b \geq dr_0 - k + l\,. \tag{17}$$

If we denote

$$\nu_i = \min_{g^{-1}(b_i)}\{\text{multiplicity of } g(z) = b_i\}\,,$$

we have $1 \leq \nu_i \leq d$. Now the number of exceptional values satisfies

$$D_g = r_0 \leq \frac{n_g + k - l}{d} = 2 + \frac{2}{R} - \frac{l}{d} \tag{18}$$

where we have used (17), hence (15) implies

$$D_g \leq 2 + \frac{2}{R} \leq 4\,.$$

In particular for algebraic minimal surfaces and its associated family, we have $R > 1$ so that

$$D_g \leq 3\,,$$

which is nothing but Osserman's theorem.

Next, we show (13). Let $b_1, \ldots, b_{l_0}$ be the *totally* ramified values which are not exceptional values. Let n_r be the sum of branching orders at $b_1, \ldots, b_{l_0}$. For each

b_i, the number of points in the inverse image $g^{-1}(b_i)$ is less than or equal to d/ν_i, since ν_i is the minimum of the multiplicity at all $g^{-1}(b_i)$. Thus we obtain

$$dl_0 - n_r \leq \sum_{i=1}^{l_0} \frac{d}{\nu_i}\,. \tag{19}$$

This implies

$$l_0 - \sum_{i=1}^{l_0} \frac{1}{\nu_i} \leq \frac{n_r}{d}\,,$$

hence using the first inequality in (16) and $n_r \leq n_b$, we get

$$\nu_g = r_0 + \sum_{i=1}^{l_0}\left(1 - \frac{1}{\nu_i}\right) \leq \frac{k+n_0}{d} + \frac{n_r}{d} \leq \frac{n_g + k}{d} = 2 + \frac{2}{R}\,.$$

The sharpness of (11) and (12) follows from:

(1) When $d = 2$ we have

$$D_g \leq 2 + \frac{2}{R} - \frac{l}{2}, \quad \nu_g \leq 2 + \frac{2}{R}\,.$$

The surface given by (8) attains both equalities, since $R = 4$, $l = 1$ and $D_g = 2$, $\nu_g = 2.5$. Thus (11) and (12) are sharp.

(2) Voss' surface satisfies $d = 1$ and $G = 0$. Thus when $k = 3$, we get $R = 2$, $l = 0$ hence $D_g = 3 = 2 + 2/2$. When $k = 4$, we have $R = 1$, $l = 0$ and $D_g = 4 = 2 + 2/1$. These show that (11) and (12) are sharp in non-algebraic pseudo-algebraic case, too.

Corollary 3.4 is obtained as follows: It is easy to see that $r_0 = 3$ implies $R \leq 2$, hence $G - 1 + \frac{1}{2}\sum_{j=1}^{k} \mu_j \leq 2(G-1) + k$. As we have $\mu_j \geq 2$ in the algebraic case, it follows

$$k \leq \frac{1}{2}\sum_{j=1}^{k} \mu_j \leq G - 1 + k\,. \tag{20}$$

Thus we obtain (i). When $G = 1$, (20) implies $\mu_j = 2$ for all j, which means that all the ends are embedded ([JM]). Therefore, if M has a non-embedded end, then $r_0 \leq 2$. When $r_0 = 3$, from (14), $d = k$ holds, and hence $R = 2$. Therefore, from (11), $l = 0$ holds, which means that g does not branch in M. □

Remark 4.1. The inequality (11) gives more informations than (10). In particular, (11) implies that the more branch points g has in M, the less is the number of exceptional values.

Remark 4.2. The inequality (10) is also best possible for algebraic minimal surfaces in the following sence. In [MS, Theorem 3], we constructed two infinite series of mutually distinct algebraic minimal surfaces of the fixed topological type $G = 1$ and $k = 4$, whose Gauss map omits 2 values. (There are some errors in

signatures in [MS, Lemma 4.1], but no effects on the result.) They are given as follows. Let $\overline{M}$ be the square torus on which the Weierstrass $\wp$ function satisfies $(\wp')^2 = 4\wp(\wp^2 - a^2)$. Let M be given by removing 4 points satisfying $\wp = 0, \pm a, \infty$ from $\overline{M}$. Define the Weierstrass data by

[Case 1] $g = \dfrac{\sigma}{\wp^j \wp'}, \quad hdz = \dfrac{\wp d\wp}{\wp'}, \quad j = 1, 2, 3 \ldots,$

[Case 2] $g = \dfrac{\sigma}{\wp^j \wp'}, \quad hdz = \dfrac{\wp^{j+1} d\wp}{\wp'}, \quad j = 2, 4, 6 \ldots,$

Then choosing a suitable σ, we obtain algebraic minimal surfaces with g omitting 2 values 0 and ∞. Since the degree of g is $d = 2j + 3$ in both cases and $R = d/2 = (2j+3)/2$, $2 + 2/R$ tends to 2 ($= D_g$) as close as we like. (Costa's surface is given by $j = 0$, in which case $(G, k, d) = (1, 3, 3)$, and g omits just one value 0.)

5. Unicity theorem and some other results

We give two applications of Theorem 3.3. The first one is an extension of Fujimoto's unicity theorem for algebraic minimal surfaces [F3] to the pseudo-algebraic case. By using Fujimoto's argument and Theorem 3.3, we obtain the following result:

Theorem 5.1. *Consider two non-flat pseudo-algebraic minimal surfaces M_1, M_2 with the same basic domain $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$. Let G be the genus of $\overline{M}$, and let g_1, g_2 be the Gauss map of M_1 and M_2 respectively. Assume that g_1 and g_2 have the same degree d when considered as a map on $\overline{M}$, but assume $g_1 \neq g_2$ as a map $M \to \mathbb{P}^1$. Let $c_1, \ldots, c_q \in \mathbb{P}^1$ be distinct points such that $g_1^{-1}(c_j) \cap M = g_2^{-1}(c_j) \cap M$ for $1 \leq j \leq q$. Then*

$$q \leq 4 + \frac{2}{R}, \quad \frac{1}{R} = \frac{G - 1 + k/2}{d} \leq 1 \tag{21}$$

follows. In particular, $q \leq 6$, and for algebraic minimal surfaces we have $q \leq 5$.

Proof. Put

$$\delta_j = \sharp(g_1^{-1}(c_j) \cap M) = \sharp(g_2^{-1}(c_j) \cap M)\,,$$

where $\sharp$ denotes the number of points. Then we have

$$qd \leq k + \sum_{j=1}^{q} \delta_j + n_g\,, \tag{22}$$

using the same notation as in §4. Consider a meromorphic function $\varphi = \dfrac{1}{g_1 - g_2}$ on M. Then at each point of $g_1^{-1}(c_j) \cap M$, φ has a pole, while the total number of the poles of φ is at most $2d$, hence we get

$$\sum_{j=1}^{q} \delta_j \leq 2d\,. \tag{23}$$

Then from (22) and (23), we obtain

$$qd \leq k + 2d + n_g\,,$$

and

$$q \leq \frac{2d + n_g + k}{d} = 4 + \frac{2}{R}$$

follows immediately. □

REMARK 5.2. Fujimoto [F3] gives an example of two pseudo-algebraic minimal surfaces with $q = 6$, of which Gauss maps do not coincide. For algebraic case, whether $q = 5$ is best possible or not is another interesting open problem.

The second application of Theorem 3.3 is a proof of Gackstätter's result [G] :

PROPOSITION 5.3 (Gackstätter [G]). *If the Gauss map of an algebraic minimal surface with $G = 1$ omits 3 values $a_1, a_2, a_3 \in \mathbb{P}^1$, then all branch points of g are located at the end points, and g is a non-branched covering map of $\mathbb{P}^1 \setminus \{a_1, a_2, a_3\}$.*

PROOF. This follows immediately from Corollary 3.4 (ii). □

Thus the Gauss map descends to $\mathbb{P}^1 \setminus \{3 \text{ points}\}$, but the minimal surface is not obtained from a covering of a minimal surface defined on $\mathbb{P}^1 \setminus \{3 \text{ points}\}$, otherwise, by (ii) of Corollary 3.4, $D_g \leq 2$. This implies that hdz can not descends to $\mathbb{P}^1 \setminus \{3 \text{ points}\}$.

We will use Proposition 5.3 in order to give a lower bound for the characteristic function $T_g(r)$ in Proposition 6.3.

The following is obvious:

PROPOSITION 5.4. *If the Gauss map g of a pseudo-algebraic minimal surface omits exactly r values $a_1, \ldots, a_r \in \mathbb{P}^1$ for $r = 3, 4$, and has no branch points in the basic domain M, then g is a non-branched covering of $\mathbb{P}^1 \setminus \{a_1, \ldots, a_r\}$.*

In this situation, the universal covering surface of M and of $\mathbb{P}^1 \setminus \{a_1, \ldots, a_r\}$ are disks, which we denote by $\mathbb{D}$ and Ω, respectively. When g has no branch points in M, the lifted map $g : \mathbb{D} \to \Omega$ is a non-branched holomorphic map, i.e., a hyperbolic isometry. Since the degree of g restricted to $\overline{M}$ is d, the fundamental domain of M is given by $\cup_{i=1}^{d} U_i \subset \mathbb{D}$, where each U_i is diffeomorphic to $\mathbb{P}^1 \setminus \{a_1, \ldots, a_r\}$.

EXAMPLE 5.5. Voss' surfaces are examples for $d = 1$.

6. Toward the Nevanlinna theory

Unfortunately, the above argument does not prove the conjecture in §3. To go further, we state some links to the Nevanlinna theory [Ko].

We consider the case where the universal covering surface of M is a unit disk $\mathbb{D}$. In order to adjust to the Nevanlinna theory, we use the hyperbolic metric ω_h with curvature -4π on $\mathbb{D}$, and the Fubini-Study metric ω_{FS} with curvature 4π on $\mathbb{P}^1$

(hence $\mathbb{P}^1$ has area 1). Then by Gauss-Bonnet's theorem for a complete punctured Riemann surface with hyperbolic metric, we have

$$2\pi\chi(M) = \int_M K_h\omega_h = -4\pi\int_M \omega_h = -4\pi A_{hyp}(M)\,, \tag{24}$$

where $A_{hyp}(M)$ is the hyperbolic area of M, hence for the fundamental domain F of M, we get

$$A_{hyp}(F) = G - 1 + \frac{k}{2}\,. \tag{25}$$

REMARK 6.1. The Gauss-Bonnet theorem (24) for (M, ω_h) is often used without proof, so here we give a brief proof. Let D_{ε_j} be the disk with radius ε_j around p_j, $j = 1, 2, \ldots, k$. We denote $M_\varepsilon = \overline{M} \setminus \cup_j D_{\varepsilon_j}$, and by $\varepsilon \to 0$, we mean all $\varepsilon_j \to 0$. Consider any metric σ on $\overline{M}$ which is flat in all D_{ε_j}. Denoting locally (as Kähler forms) $\sigma = \dfrac{i}{2}\tilde{\sigma}dz \wedge d\bar{z}$ and $\omega_h = \dfrac{i}{2}\tilde{\omega}_h dz \wedge d\bar{z}$, we have by Stokes' theorem

$$-\sum_j \int_{\partial D_{\varepsilon_j}} d^c \log(\sigma/\omega_h) = \int_{M_\varepsilon} dd^c \log(\sigma/\omega_h) = \int_{M_\varepsilon} dd^c \log\tilde{\sigma} - \int_{M_\varepsilon} dd^c \log\tilde{\omega}_h\,,$$

where $d = \partial + \bar{\partial}$, $d^c = (\partial - \bar{\partial})/(4\pi i)$, (here ∂ is the half of Osserman's one). Because $dd^c \log\tilde{\omega} = -\dfrac{K_\omega}{2\pi}dA_\omega$ holds where K_ω and dA_ω are the Gauss curvature and the area form of ω, respectively, taking the limit $\varepsilon \to 0$ and applying the Gauss-Bonnet's theorem to $(\overline{M}, \sigma)$, we obtain

$$\lim_{\varepsilon\to 0}\int_{M_\varepsilon} (dd^c \log\tilde{\sigma} - dd^c\log\tilde{\omega}_h) = -\chi(\overline{M}) - 2A_{hyp}(M)\,.$$

Next, take a local coordinate on each D_{ε_j} so that $z = 0$ corresponds to p_j. Then we can express $\sigma = \dfrac{i}{2}dz \wedge d\bar{z}$ and $\omega_h = \dfrac{i}{2\pi}\dfrac{dz \wedge d\bar{z}}{|z|^2(\log|z|^{-2})^2}$ on D_{ε_j}. Noting that $d^c = \dfrac{1}{4\pi}\big(-\dfrac{1}{r}\dfrac{\partial}{\partial\theta}dr + r\dfrac{\partial}{\partial r}d\theta\big)$, we obtain

$$\lim_{\varepsilon\to 0}\sum_j \int_{\partial D_{\varepsilon_j}} d^c\log(\sigma/\omega_h) = k\,,$$

which implies (24) and (25). □

Next, let d be the degree of g, then the area $A_{FS}(F)$ of F with respect to the induced metric $g^*\omega_{FS}$ is d. Thus we obtain

$$A_{FS}(F) = \frac{d}{G - 1 + k/2}A_{hyp}(F) = RA_{hyp}(F)\,. \tag{26}$$

We now know the meaning of the ratio R; the ratio of the area of the fundamental domain with respect to the induced Fubini-Study metric to the one with respect to the hyperbolic metric on $\mathbb{D}$.

REMARK 6.2. Even when the conformal type of M is not hyperbolic, the ratio R is meaningful in Theorem 3.3.

Now, remember Shimizu-Ahlfors' theorem on the characteristic function $T_g(r)$ of g, in terms of

$$T_g(r) = \int_0^r \frac{dt}{t} \int_{\mathbb{C}(t)} g^* \omega_{FS} \,.$$

Here $\mathbb{C}(t)$ is the subdisk of $\mathbb{D}$ with radius $0 < t < 1$. In order to develop the Nevanlinna theory on meromorphic functions on the unit disk, we need the growth order of $T_g(r)$ compared with

$$\int_0^r \frac{dt}{t} \int_{\mathbb{C}(t)} \omega_h \approx \frac{1}{2} \log \frac{1}{1-r} \,,$$

where r is sufficiently close to 1 (strictly, the left hand side is $\frac{1}{2} \log \frac{1}{1-r^2}$). We always use this approximation formula in the following, because in the Nevanlinna theory, a bounded quantity is ignored.

Actually, we want to know the best estimate of type

$$T_g(r) \geq \frac{\eta}{2} \log \frac{1}{1-r} \,, \tag{27}$$

to get the lemma on logarithmic derivatives on meromorphic functions on the unit disk [KKM]. In general, the area of $\mathbb{C}(t)$ is approximated by that of finite union of fundamental domains $\cup F_j$. It seems that we get η from (26), but we need some argument here, since the hyperbolic symmetry never fits the shape of the disk. We do not go into details of this argument, instead, we give some examples in easier cases.

If we replace the Fubini-Study metric by a singular metric Ψ on $\mathbb{P}^1$ with area 1, we have

$$T_g(r) \geq \int_0^r \frac{dt}{t} \int_{\mathbb{C}(t)} g^* \Psi \,. \tag{28}$$

This is shown rather easily by using Crofton's formula in the integral geometry [Ko]. When the image $g(M)$ is $\mathbb{P}^1 \setminus \{r \text{ points}\}$, where $r = 3$ or 4, the singular metric Ψ on $\mathbb{P}^1$ induced by the hyperbolic metric on Ω normalized so that the area of $g(M)$ (counted without multiplicity) is 1 fits the case. Using this metric, we give a few computable examples.

Proposition 6.3. *Consider a pseudo-algebraic minimal surface with the basic domain $M = \overline{M} \setminus \{p_1, \ldots, p_k\}$, and assume that g branches only at p_j's.*

(i) *If $D_g = 3$, we have*

$$T_g(r) \geq \log \frac{1}{1-r} \,. \tag{29}$$

This is satisfied by Voss' surface with $k = 3$, and an algebraic minimal surface with $G = 1$ and $D_g = 3$, if any.

(ii) *If* $D_g = 4$, *we have*

$$T_g(r) \geq \frac{1}{2} \log \frac{1}{1-r} \,. \tag{30}$$

This is satisfied by Voss' surface with $k = 4$.

Remark 6.4. For some reasons, we conjecture that (29) and (30) are equalities, and $\eta = 2$ in (i) and $\eta = 1$ in (ii) hold, which implies $\eta = R$ in these cases.

Proof. Let $\mathbb{D}$ be the universal covering disk of M, and Ω that of $\mathbb{P}^1 \setminus \{a_1, \ldots, a_{r_0}\}$, where $a_1, \ldots, a_{r_0}$ are the exceptional values of g. Let ω_h and ω_Ω be the hyperbolic metric with curvature -4π. Denote by $g : \mathbb{D} \to \Omega$ the lifted Gauss map. Since this is not branched, g is a hyperbolic isometry. To obtain the characteristic function $T_g(r)$, normalize the metric ω_Ω so that the fundamental domain of $\mathbb{P}^1 \setminus \{a_1, \ldots, a_{r_0}\}$ has area 1. When $D_g = r_0 = 3$, this area with respect to ω_Ω is $G - 1 + 3/2 = 1/2$ by (25), thus we use the metric $2\omega_\Omega$ in (28), and we get

$$\begin{aligned} T_g(r) &\geq \int_0^r \frac{dt}{t} \int_{\mathbb{C}(t)} 2g^*\omega_\Omega = 2 \int_0^r \frac{dt}{t} \int_{\mathbb{C}(t)} \omega_h \\ &= \log \frac{1}{1-r} \,. \end{aligned}$$

The last assertion in (i) follows from Proposision 5.3. When $D_g = r_0 = 4$, we need no change of the metric, and get (ii). □

What we have to do is, however, the converse. That is, to estimate $T_g(r)$ first, and then get a bound for D_g or ν_g. We leave this to [KKM].

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